

Generalized Eilenberger theory for Majorana zero-mode-carrying disordered p -wave superconductors

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Disorder is known to suppress the gap of a topological superconducting state that would support non-Abelian Majorana zero modes. In this paper, we study using the self-consistent Born approximation the robustness of the Majorana modes to disorder within a suitably extended Eilenberger theory, in which the spatial dependence of the localized Majorana wave functions is included. We find that the Majorana mode becomes delocalized with increasing disorder strength as the topological superconducting gap is suppressed. However, surprisingly, the zero bias peak seems to survive even for disorder strength exceeding the critical value necessary for closing the superconducting gap within the Born approximation.

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I. INTRODUCTION

Recently, considerable theoretical effort has been put into the search for localized non-Abelian Majorana modes (MMs) in solid state systems [1–4]. All proposed systems for the realization of MMs rely on the presence of p -wave superconductivity, either intrinsically [5] or artificially through a careful engineering of heterostructure design [6–11]. The theoretical progress in this field has sparked substantial experimental effort in realizing the proposed systems [12–16], with several measurements reporting the observation of the theoretically predicted [8–10] zero bias peaks (ZBPs) in conductance measurements in semiconductor nanowires consistent with the existence of zero energy MMs.

However, a compelling and unambiguous signature for the MMs is still lacking. Among various complications, an unfavorable factor in experiments is the disorder invariably present in all real experimental samples. With the possible exception of topological insulator-based heterostructures [6, 7], in which the effects of disorder are minimal [17], other semiconductor-based heterostructures [8, 10, 11] and even the idealized p -wave superconductor [5] are all susceptible to disorder, since the Anderson theorem asserting the insensitivity of superconductivity to ordinary spin-independent momentum scattering does not in general apply to p -wave superconducting ordering.

The effects of disorder in such topological systems have been previously investigated [18–31] from a number of different perspectives. One approach to the problem consists of introducing many realizations of disorder and ensemble-averaging at the end to extract universal properties [23, 25, 27, 28, 30, 31]. While this approach is more akin to the experimental situation (where there is only a single realization of disorder at each setup), the end

result of the posterior disorder averaging is mostly numerical and few analytical statements can be made. On the other hand, previous attempts of anterior disorder averaging were mostly concerned with the properties in the bulk [18–22, 24, 26]. The effects of ensemble-averaged disorder on the end MMs were not fully investigated.

In this paper, we undertake the task of analyzing the effects of ensemble-averaged disorder on a topological one-dimensional (1D) system, the idealized spinless p -wave superconducting wire. In particular, we treat the disorder in the self-consistent Born approximation (SCBA) and investigate its effects on the spectral properties of the whole system, with an emphasis on its boundary where the MMs reside. We are generalizing in the current work the earlier work [26] done by two of the authors in order to analytically assess the disorder effects on the MMs themselves, not just on the topological superconducting gap. Of course, this problem has been much studied in the recent MM literature, but most of this disorder work is carried out purely numerically—a typical example being our own work presented in Ref. 28. The goal in the current work is to develop an analytical approach to the problem just as we did for the superconducting gap in Ref. 26.

To generalize SCBA to inhomogeneous structures, it is convenient to adopt a formalism similar to the Eilenberger equations [32], but including pair-breaking effects of disorder in the theory. Our formalism differs from the conventional quasiclassical treatment of disordered superconductors in two ways. First, unlike the conventional approach, we consider only weak disorder and do not take the diffusive limit to derive the Usadel equations [33], as this would wipe out the spectral gap in the topological system. It is important to emphasize that the standard Usadel formalism for ordinary metallic superconductivity, which is the generalization of the Eilenberger theory to include disorder in the diffusive limit, is inapplicable to the topological superconducting situation of interest here since the system becomes gapless in the Usadel limit. Our current work provides the appropriate

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disorder generalization of the quasiclassical Eilenberger theory for the disordered p -wave topological (i.e. MM-carrying) superconducting system. The standard Usadel theory cannot be used for topological superconductors since the implicit assumption of strong disorder scattering in the Usadel theory makes it inapplicable to the effective p -wave superconducting pairing underlying the MM-carrying topological systems. The second difference of our work from the conventional treatment is that we do not start by integrating out the fast-oscillating parts of the Green function, but instead consider the Green function in a chiral basis and keep all its spatial dependence. This is possible only for 1D problems, and is essential to extract the exact spatial dependence of the MM.

We emphasize that the focus of this paper is on the semiconductor heterostructures in the presence of spin-orbit coupling and spin splitting [6–11] proposed to realize MMs, all of which have s -wave pairing terms induced by proximity effect through Cooper pairs tunneling from proximate superconductors. By projecting the Hamiltonians of these systems to their low-energy subspaces [10, 34, 35], one universally obtains effective p -wave superconductors but with model-specific pairings and scattering strengths [see Eq. (1) below]. The crucial difference of this semiconductor Majorana nanowire from an intrinsic p -wave superconducting wire is that, since now the pairing term is proximity-induced, it is both unnecessary and inappropriate to perform self-consistent theoretical calculations because there is no intrinsic pairing interaction in the wire itself—the pairing is induced entirely from the outside through the proximity effect [36]. This issue has already been discussed in previous numerical [37] or quasiclassical [38] investigations of these systems, and we shall thus take the pairing strength as a fixed parameter without solving the self-consistent gap equation following all earlier theoretical works in the literature on this problem.

The paper is organized as follows. In Sec. II we briefly discuss the formalism we adopt to analyze our current p -wave superconducting system. Then we compare this approach with the result obtained from SCBA in the bulk in Sec. III. Next the effect of disorder on the MMs at the boundary is investigated in Sec. IV, where the Eilenberger equations are solved for a semi-infinite 1D system. In Sec. VI we discuss the manifestation of hybridization between the MM and the continuum modes in this formalism. Finally, in Sec. VII we summarize our results.

II. FORMALISM

We consider a semi-infinite wire at $x > 0$ described by the linearized Hamiltonian

$$H = \sum_{C=R/L} \int_0^\infty dx \left[-iv_F s_C \psi_C^\dagger \partial_x \psi_C + \Delta s_C \psi_C \psi_{\bar{C}} + V_f \psi_C^\dagger \psi_C + V_b \psi_C^\dagger \psi_{\bar{C}} \right]. \quad (1)$$

Here v_F is the Fermi velocity, Δ is the p -wave superconducting order parameter, and $s_C = \pm 1$ for $C = R/L$, where R/L denotes right/left moving electrons. $V_{f/b}$ is the forward/backward scatterings due to static quenched disorder, assumed to be short ranged in this work. Coulomb disorder, which might be present in real semiconductor nanowire systems of experimental interest, will typically be screened by the surrounding gates, the normal leads, the superconductor, and by the electrons in the wire themselves leading presumably to short-ranged elastic disorder. The linearized form of disorder in Eq. (1) is related to the full disorder potential U by

$$V_f(x) = \sum_{q \sim 0} U_q e^{iqx}, \quad (2)$$

$$V_b(x) = \sum_{q \sim 0} U_{q-2k_F} e^{iqx}, \quad (3)$$

with k_F being the Fermi momentum.

The spectral properties of the system are encoded in the Nambu-Gorkov Green function $G(x, t, x', t') = -i \langle \mathcal{T} \Psi(x, t) \Psi^\dagger(x', t') \rangle$, where $\Psi = (\psi_R, \psi_L, \psi_L^\dagger, \psi_R^\dagger)^T$. We are interested, following the spirit of the Eilenberger theory which is being generalized in the current work, in the Green function defined as

$$g(x, \omega) = v_F i \lim_{\epsilon \rightarrow 0^+} [G(x, x - \epsilon) + G(x, x + \epsilon)] \sigma_3 \tau_3 \quad (4)$$

where σ and τ are Pauli matrices acting on the R/L space and particle-hole space, respectively. To extract the density of states (DOS) from g , we note that since the fermion operator is linearized in the form of $\psi(x) \simeq \psi_R e^{ik_F x} + \psi_L e^{-ik_F x}$, the Green function of $\psi(x)$ is related to the Green function in the chiral basis via

$$G^{(0)}(x, x') \simeq G_{RR} e^{ik_F(x-x')} + G_{RL} e^{ik_F(x+x')} + G_{LR} e^{-ik_F(x+x')} + G_{LL} e^{-ik_F(x-x')}. \quad (5)$$

Therefore, the DOS is given by

$$\nu(x, \omega) = \frac{\nu_0}{4} [\text{TrRe}(g \sigma_3 \tau_3) - \text{TrRe}(g \sigma_- \tau_3) e^{2ik_F x} + \text{TrRe}(g \sigma_+ \tau_3) e^{-2ik_F x}], \quad (6)$$

$$= \frac{\nu_0}{4} \text{TrRe}(g \sigma_3 \tau_3 + 2g \sigma_+ \tau_3 \cos 2k_F x), \quad (7)$$

where $\nu_0 = \frac{1}{\pi v_F}$ is the DOS in the normal state and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. In Eq. (7), the first(second) term contains the slowly (fast) -oscillating part of the DOS. Conventional derivation of Eilenberger equations [32, 38] effectively ignores the second term. One key aspect of our generalization is keeping these oscillatory terms which can be done completely analytically (at least for the 1D problem of current interest).

The equation of motion of g can be derived from the Dyson's equations of G . As both spatial arguments of G are set to x , we must use the two conjugate Dyson's equations:

$$(\omega - H_{\text{BdG}} - \Sigma) G(x, y) = \delta(x - y), \quad (8a)$$

$$G(y, x) (\omega - H_{\text{BdG}} - \Sigma) = \delta(x - y), \quad (8b)$$

where $H_{\text{BdG}} = -iv_F\sigma_3\tau_3\partial_x + \Delta\sigma_3\tau_1$ and Σ is the self-energy due to ensemble-averaged disorder V_f and V_b in Eq. (1). Here the derivative acting on the right is understood as $G(y, x) i\overleftarrow{\partial}_x = -i\partial_x G(y, x)$. By collecting the terms $\partial_x G(x, y)$ and $\partial_x G(y, x)$, we have

$$v_F\partial_x g = i[\omega\sigma_3\tau_3 - i\Delta\tau_2 - \sigma_3\tau_3\Sigma, g]. \quad (9)$$

The derivation of Σ in the SCBA is found to be (see Appendix. A)

$$\begin{aligned} \Sigma(x) &= D_f\tau_3G\tau_3 \\ &+ \frac{D_b}{2}(\sigma_1\tau_3G\sigma_1\tau_3 + \sigma_2\tau_3G\sigma_2\tau_3), \end{aligned} \quad (10)$$

where the disorder strengths D_f and D_b are defined by

$$\langle V_f(x) V_f(x') \rangle = D_f \delta(x - x'), \quad (11a)$$

$$\langle V_b(x) V_b(x') \rangle = 0, \quad (11b)$$

$$\langle V_b(x) V_b^*(x') \rangle = D_b \delta(x - x'). \quad (11c)$$

We now resolve Eq. (9) into components with the observation that in the bulk of a clean system g is exactly known to be

$$g_{\text{bulk}} = \frac{-i\omega}{\sqrt{\Delta^2 - \omega^2}} \sigma_3\tau_3 - \frac{\Delta}{\sqrt{\Delta^2 - \omega^2}} \sigma_0\tau_2. \quad (12)$$

Consider now a situation where $D_{f/b}$ are adiabatically tuned away from zero in the bulk of the wire. By substituting Eq. (12) in Eq. (9), it can be shown that g can only have six non-zero components:

$$\begin{aligned} g &= g_{31}\sigma_3\tau_1 + g_{02}\sigma_0\tau_2 + g_{33}\sigma_3\tau_3 \\ &+ g_{10}\sigma_1\tau_0 + g_{21}\sigma_2\tau_1 + g_{23}\sigma_2\tau_3, \end{aligned} \quad (13)$$

and their equations of motions are

$$v_F\partial_x g_{31} = 2\omega g_{02} + 2i\Delta g_{33} + \frac{2i}{\tau} g_{02}g_{33}, \quad (14a)$$

$$v_F\partial_x g_{02} = -2\omega g_{31}, \quad (14b)$$

$$v_F\partial_x g_{33} = -2i\Delta g_{31} - \frac{2i}{\tau} g_{31}g_{02}, \quad (14c)$$

$$v_F\partial_x g_{10} = 2\omega g_{23} - \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}}\right) (g_{21}g_{31} + g_{23}g_{33}), \quad (14d)$$

$$v_F\partial_x g_{21} = 2i\Delta g_{23} + \left(\frac{3i}{\tau} - \frac{4i}{\tilde{\tau}}\right) g_{02}g_{23} + \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}}\right) g_{10}g_{31}, \quad (14e)$$

$$v_F\partial_x g_{23} = -2\omega g_{10} - 2i\Delta g_{21} - \left(\frac{3i}{\tau} - \frac{4i}{\tilde{\tau}}\right) g_{02}g_{21} + \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}}\right) g_{10}g_{33}, \quad (14f)$$

where we have defined $\tau^{-1} = \pi\nu_0 D_b$ and $\tilde{\tau}^{-1} = \pi\nu_0 \frac{1}{2}(D_f + D_b)$. Substituting Eq. (13) in Eq. (7), we have for the DOS

$$\nu(x, \omega) = \nu_0 (\text{Re}g_{33} - \text{Im}g_{23} \cos 2k_F x). \quad (15)$$

To completely formulate the problem, Eqs. (14) must be supplemented with boundary conditions. In the bulk of the wire ($x \rightarrow \infty$), since the BdG Hamiltonian is diagonal in the σ space, the resultant Green function must also be diagonal in the σ -space. This implies that $g_{10} = g_{21} = g_{23} = 0$ at $x \rightarrow \infty$. By setting the spatial derivatives of Eq. (14) to zero, we also obtain

$$\omega g_{02} + i\Delta g_{33} + \frac{i}{\tau} g_{02}g_{33} = 0, \quad (16a)$$

$$g_{31} = 0, \quad (16b)$$

$$-i\Delta g_{31} - \frac{i}{\tau} g_{31}g_{02} = 0 \quad (16c)$$

$$g_{02} = 0, \quad (18a)$$

$$g_{10} = 1, \quad (18b)$$

$$g_{21} = ig_{31}, \quad (18c)$$

$$g_{23} = ig_{33} \quad (18d)$$

at $x \rightarrow \infty$.

To derive the boundary conditions at the end of the wire ($x = 0$), we note that since the fermion operator is linearized as $\psi(x) = \psi_R(x) e^{ik_F x} + \psi_L(x) e^{-ik_F x}$, at the end of wire we have $0 = \psi(0) = \psi_R(0) + \psi_L(0)$. This translates to the requirement that

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} G(0, \epsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (17)$$

Since it follows from the definition of g [Eq. (4)] and the Dyson's equation for G [Eq. (8)] that $\lim_{\epsilon \rightarrow 0^+} G(0, \epsilon) = \frac{1}{2iv_F} g(0) \sigma_3\tau_3 + \frac{i}{2v_F} \sigma_3\tau_3$, we have

at $x = 0$. The last condition is also consistent with the requirement that $\nu(0, \omega) = 0$ [c.f. Eq. (15)].

Finally we add that since $g^2 = 1$ in the bulk of a clean system [c.f. Eq. (12)] and from Eq. (9) we have $\partial_x g^2 = 0$, the normalization $g^2 = 1$ is valid throughout the whole system. Written in its components,

$$g_{31}^2 + g_{02}^2 + g_{33}^2 + g_{10}^2 + g_{21}^2 + g_{23}^2 = 1. \quad (19)$$

We make two remarks before closing the discussion on the formalism. First, note that Eqs. (14a)-(14c) do not contain the variables g_{10} , g_{21} , and g_{23} . Together with the boundary conditions Eqs. (16) and Eq. (18a), g_{31} , g_{02} , and g_{33} can thus be solved without reference to the other three variables. These equations have been previously derived [32, 39] by first integrating out the fast-oscillating degrees of freedom in the problem, or equivalently [see Eq. (5)] by assuming that G is always diagonal in σ -space. We have seen from Eqs. (18) that this cannot hold true near the boundary, where the reflection from the end of the wire induces correlations between left- and right-moving modes. For our current work, keeping these oscillatory terms, which are always neglected in the usual Eilenberger theory, is crucial since our interest is in figuring out the effect of disorder on the MMs which reside at the boundaries (i.e., at the wire ends of the 1D system).

It can be seen from Eq. (15) that computation of DOS

using g_{33} alone would miss spatially rapid oscillations near the end of the wire. Indeed, it has been pointed out in Ref. 38 that with the reduced set of variables $\{g_{31}, g_{02}, g_{33}\}$, an oscillatory factor ($\propto \cos 2k_F x$) of the DOS near the end of the wire is not captured. It is therefore necessary to solve the whole set of equations (14) if a spatial resolution of the DOS under the Fermi wavelength is desired. However, in the following sections in this paper, we shall only focus on $\{g_{31}, g_{02}, g_{33}\}$ for simplicity.

Lastly we adopt this formalism to the case of conventional s -wave superconductivity, with the linearized Hamiltonian

$$H_0 = \sum_{C, \sigma} \int_0^\infty dx \left[-iv_F s_C \psi_{C\sigma}^\dagger \partial_x \psi_{C\sigma} + \Delta_s \psi_{C\sigma} \psi_{\bar{C}, \bar{\sigma}} + V_f \psi_{C\sigma}^\dagger \psi_{C\sigma} + V_b \psi_{C\sigma}^\dagger \psi_{\bar{C}, \sigma} \right], \quad (20)$$

where only non-magnetic disorder $V_{f/b}$ is considered here. Repeating the above procedures in solving for $\partial_x g^{(s)}$ and then decomposing $g_s^{(s)}$ as

$$g^{(s)} = g_{01}^{(s)} \sigma_0 \tau_1 + g_{32}^{(s)} \sigma_3 \tau_2 + g_{33}^{(s)} \sigma_3 \tau_3 + g_{10}^{(s)} \sigma_1 \tau_0 + g_{22}^{(s)} \sigma_2 \tau_2 + g_{23}^{(s)} \sigma_2 \tau_3, \quad (21)$$

we reach the following set of differential equations:

$$v_F \partial_x g_{01}^{(s)} = 2\omega g_{32}^{(s)} + 2i\Delta_s g_{33}^{(s)}, \quad (22a)$$

$$v_F \partial_x g_{32}^{(s)} = -2\omega g_{01}^{(s)} - \frac{2i}{\tau} g_{01}^{(s)} g_{33}^{(s)}, \quad (22b)$$

$$v_F \partial_x g_{33}^{(s)} = -2i\Delta_s g_{01}^{(s)} + \frac{2i}{\tau} g_{01}^{(s)} g_{32}^{(s)}, \quad (22c)$$

$$v_F \partial_x g_{10}^{(s)} = 2\omega g_{23}^{(s)} - 2i\Delta_s g_{22}^{(s)} - \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}} \right) \left(g_{22}^{(s)} g_{32}^{(s)} + g_{23}^{(s)} g_{33}^{(s)} \right), \quad (22d)$$

$$v_F \partial_x g_{22}^{(s)} = 2i\Delta_s g_{10}^{(s)} - \left(\frac{3i}{\tau} - \frac{4i}{\tilde{\tau}} \right) g_{01}^{(s)} g_{23}^{(s)} + \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}} \right) g_{10}^{(s)} g_{32}^{(s)}, \quad (22e)$$

$$v_F \partial_x g_{23}^{(s)} = -2\omega g_{10}^{(s)} + \left(\frac{3i}{\tau} - \frac{4i}{\tilde{\tau}} \right) g_{01}^{(s)} g_{22}^{(s)} + \left(\frac{i}{\tau} - \frac{4i}{\tilde{\tau}} \right) g_{10}^{(s)} g_{33}^{(s)}, \quad (22f)$$

and the boundary conditions that

$$g_{10}^{(s)} = g_{22}^{(s)} = g_{23}^{(s)} = 0, \quad (23a)$$

$$\omega g_{32}^{(s)} + i\Delta_s g_{33}^{(s)} = 0, \quad (23b)$$

$$-\omega g_{01}^{(s)} - \frac{i}{\tau} g_{01}^{(s)} g_{33}^{(s)} = 0, \quad (23c)$$

$$-i\Delta_s g_{01}^{(s)} + \frac{i}{\tau} g_{01}^{(s)} g_{32}^{(s)} = 0 \quad (23d)$$

at $x \rightarrow \infty$ and

$$g_{01}^{(s)} = 0, \quad (24a)$$

$$g_{10}^{(s)} = 1, \quad (24b)$$

$$g_{22}^{(s)} = ig_{32}^{(s)}, \quad (24c)$$

$$g_{23}^{(s)} = ig_{33}^{(s)} \quad (24d)$$

at $x = 0$. A normalization condition similar to Eq. (19) also holds:

$$\left(g_{01}^{(s)} \right)^2 + \left(g_{32}^{(s)} \right)^2 + \left(g_{33}^{(s)} \right)^2 + \left(g_{10}^{(s)} \right)^2 + \left(g_{22}^{(s)} \right)^2 + \left(g_{23}^{(s)} \right)^2 = 1. \quad (25)$$

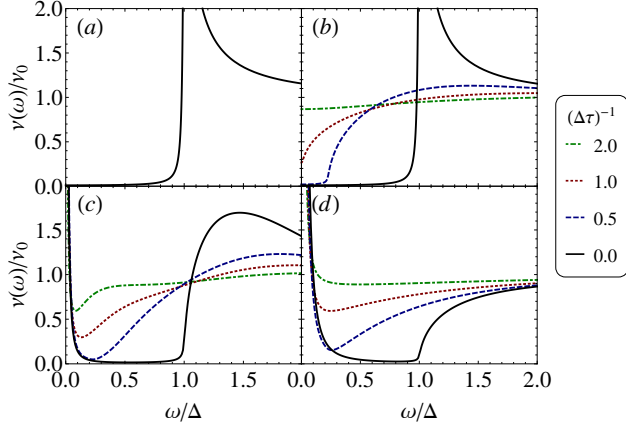


Figure 1. (Color online) (a) DOS for a semi-infinite s -wave superconducting wire. Note the result is position-independent, and is not affected by disorder. (b)-(d) The DOS of a semi-infinite p -wave superconducting wire, from the clean limit $\tau^{-1} = 0$ to a heavily disordered case $\tau^{-1} = 2\Delta$ at (b) $x \rightarrow \infty$ (in the bulk), (c) $x = \xi_0$, and (d) $x = 0$, respectively. For all plots, the energy spectra are broadened by $\eta = 0.01\Delta$.

We observe that $g_{01}^{(s)}$, $g_{32}^{(s)}$, and $g_{33}^{(s)}$ can be solved from Eqs. (22a)-(22c), Eqs. (23b)-(23d) and Eq. (24a) independent of the remaining components.

III. DOS IN THE BULK

Equations (14) can be understood as a generalization of the SCBA to spatially inhomogeneous structures. Before we utilize it to investigate into such structures, however, it is instructive to show that our formalism in the bulk indeed reduces to the SCBA result obtained earlier by two of the authors [26].

We first consider the simpler case of an s -wave superconducting wire, for which Eqs. (23) and Eq. (25) are solved by

$$g_{01}^{(s)} = 0, \quad (26a)$$

$$g_{32}^{(s)} = \frac{-\Delta_s}{\sqrt{\Delta_s^2 - \omega^2}}, \quad (26b)$$

$$g_{33}^{(s)} = \frac{-i\omega}{\sqrt{\Delta_s^2 - \omega^2}}, \quad (26c)$$

independent of the disorder parameter τ . Therefore, in the s -wave case, the DOS in the bulk is

$$\nu_s(\omega) = \nu_0 \text{Re} \left[g_{33}^{(s)}(\omega) \right] = \nu_0 \frac{\omega}{\sqrt{\omega^2 - \Delta_s^2}} \theta(\omega - \Delta_s), \quad (27)$$

plotted in Fig. 1(a), and is unaffected by disorder as required by Anderson's theorem [40].

For the case of p -wave superconducting wire in which Anderson's theorem is not applicable, a suppression of

the gap by disorder is expected. To show this, note that Eqs. (16) and Eq. (19) are solved by

$$g_{31} = 0, \quad (28a)$$

$$g_{02} = \frac{-\Delta}{\sqrt{\Delta^2 - \tilde{\omega}^2}}, \quad (28b)$$

$$g_{33} = \frac{-i\tilde{\omega}}{\sqrt{\Delta^2 - \tilde{\omega}^2}}, \quad (28c)$$

where $\tilde{\omega}$ satisfies $\tilde{\omega} = \omega + \frac{i\tilde{\omega}}{\tau\sqrt{\tilde{\omega}^2 - \Delta^2}}$. This is seen to be identical to the SCBA result of $\tilde{\omega} = \omega + (D_f + D_b)\pi\nu_0 \frac{i\tilde{\omega}}{\sqrt{\tilde{\omega}^2 - \Delta^2}}$, by noting that for point scatterers $D_f = D_b$. Figure 1(b) is a plot of the DOS evaluated by Eq. (15), for a number of disorder strengths. The bulk gap is seen to close at about $(\Delta\tau)^{-1} = 1$. In fact, it can be shown that Eq. (28) results in a degradation of the spectral gap in the form of [40] $E_{\text{gap}} = \Delta \left[1 - (\Delta\tau)^{-2/3} \right]^{3/2}$, and eventually destroys the gap for $\tau^{-1} > \Delta$. The influence of this effect on the MM located at the boundary of the wire is the focus of the following sections.

IV. DOS NEAR THE END OF THE WIRE

We now investigate the effect of ensemble-averaged disorder on the DOS near the boundary $x = 0$. Before considering the case of p -wave superconducting wire in which a MM is present, for the sake of comparison and illustration, we first review the case of a conventional s -wave superconducting wire in the current formalism. We note that the solution in the bulk given by Eq. (26) already satisfies the boundary conditions at the end of the wire [Eq. (24)]. Therefore, the DOS is uniform throughout the whole wire, and Fig. 1(a) is independent of the distance from the boundary. Thus, as expected, the boundaries of the 1D system or the wire ends do not produce any nontrivial effects for s -wave superconducting wires.

In the more nontrivial case of p -wave superconductor, the solution in the bulk Eq. (28) cannot satisfy the boundary condition at the end [Eq. (18a)] and thus Eqs. (14) must be solved directly. Without disorder, the solution is [38]

$$g_{31} = \frac{\Delta e^{-2x\sqrt{\Delta^2 - \omega^2}/v_F}}{\omega}, \quad (29)$$

$$g_{02} = \frac{\Delta \left(e^{-2x\sqrt{\Delta^2 - \omega^2}/v_F} - 1 \right)}{\sqrt{\Delta^2 - \omega^2}}, \quad (30)$$

$$g_{33} = i \frac{\Delta^2 e^{-2x\sqrt{\Delta^2 - \omega^2}/v_F} - \omega^2}{\omega\sqrt{\Delta^2 - \omega^2}}, \quad (31)$$

and the other components of g can also be solved analytically but we shall not state them here as we are ignoring variations in the length scale of k_F^{-1} . Note that

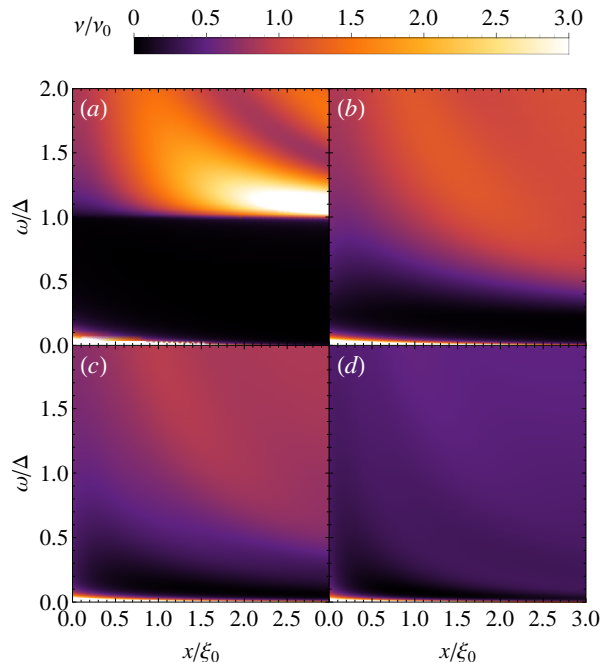


Figure 2. (Color online) The DOS $\nu(x, \omega, \eta) = \nu_0 \text{Re}[g_{33}(x, \omega + i\eta)]$ plotted as a function of position x (in units of $\xi_0 = v_F/\Delta$) and energy ω (in units of Δ), where x is measured from the end of the wire and $\eta = 0.01\Delta$ is the broadening parameter. The four panels correspond to disorder strengths (a) $\tau^{-1} = 0$, (b) $\tau^{-1} = 0.5\Delta$, (c) $\tau^{-1} = \Delta$, and (d) $\tau^{-1} = 2\Delta$. When the system is clean, the salient features are the zero-energy peak localized at the end and a pristine bulk gap. As disorder is introduced, the bulk gap shrinks and the singularity is smeared out, homogenizing the DOS of the whole system, but the zero-energy peak at the end of the wire is still visible even at strong disorder.

g_{31} is odd in frequency, indicating an odd-frequency s -wave pairing present near the boundary [38]. The close relation between the odd-frequency pairing and MMs has been emphasized in the literature [41, 42].

With nonzero disorder, the problem must be solved numerically. Figures 1(b)-1(d) show the DOS given by Eq. (15), evaluated in the bulk, at $x = \xi_0$ and $x = 0$ for a number of disorder strengths. For the same choice of disorder strengths, the contour plots of the DOS are shown in Fig. 2. In a clean wire, a singularity in DOS is present at the gap edge ($\omega = \Delta$). This singularity is absent at the end of the wire, where instead a single zero-energy MM is present. As disorder is introduced, the DOS throughout the system is homogenized, with the DOS singularity smoothed and the bulk gap suppressed. As the disorder strength is increased beyond the bulk-gap closing point of $\tau^{-1} = \Delta$, the continuum states begin to hybridize with the MM, but the ZBP is distinctly visible even under strong disorder of $\tau^{-1} = 2\Delta$, where in the bulk the DOS becomes almost flat. It might be of interest to note that at strong disorder a suppression of the DOS at $\omega \gtrsim 0$ is present only at $x \sim \xi_0$, but is absent either in the bulk or at the end of the wire. This

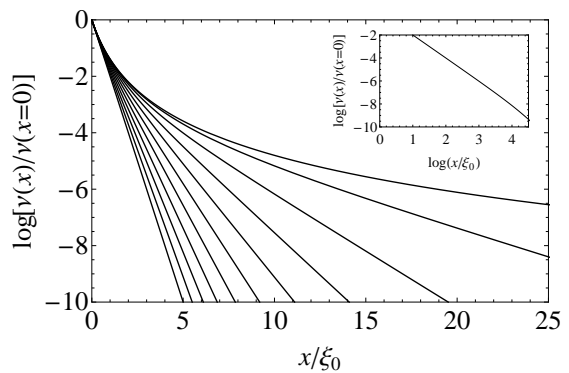


Figure 3. Log-linear plot of zero-energy DOS $\nu(x, \omega = 0)$ as a function of distance x measured from the end of a p -wave superconducting wire, with oscillations of length scale k_F^{-1} ignored. The steepest line corresponds to clean case $\tau^{-1} = 0$ where the MM is most localized. The least steep line corresponds to the critical disorder strength $\tau^{-1} = \Delta$ where the bulk gap closes. The intermediate lines are sampled at equally spaced τ^{-1} with a step size of $\delta(\tau^{-1}) = 0.1\Delta$. The inset shows the last curve corresponding to $\tau^{-1} = \Delta$ in log-log scale. Its slope is approximately -2 .

can be understood as the MM is centered at the end, its hybridization with the continuum states is the strongest there too.

We point out as an aside that the somewhat surprising continued survival of the zero mode even beyond the disorder-induced gap closing point obtained in our current formal semiclassical theory has also been seen in the direct numerical simulations carried out by two of us recently [28]. This indicates that the end MMs are very robust and exist even in the gapless p -wave superconducting phase, which might be consistent with the experimental observations where the ZBP exists even when there is no obvious gap signature in the tunneling spectrum.

V. CHANGE OF MAJORANA LOCALIZATION LENGTH UNDER DISORDER

In a clean system the MM is exponentially localized with a decay length equal to the coherence length $l_{\text{loc}} = \xi_0 = v_F/\Delta$. One expects disorder to modify this localization length, which should diverge as disorder destroys the topological phase[18]. On the one hand, the suppression of the spectral gap seems to suggest a longer decay length if it is substituted into the formula $l_{\text{loc}} = v_F/E_{\text{gap}}$. On the other hand, in the case of s -wave superconductors, the coherence length of a strongly disordered system is shortened to $\xi_{\text{dis}} \approx v_F\sqrt{\tau/\Delta}$, which suggests a shorter decay length if the formula $l_{\text{loc}} = \xi_{\text{dis}}$ is to be trusted. Equations (14) allow for a quantitative investigation of the problem.

The decay length is extracted in the following way. The DOS is related to the Green function by $\nu(x, \omega) \propto$

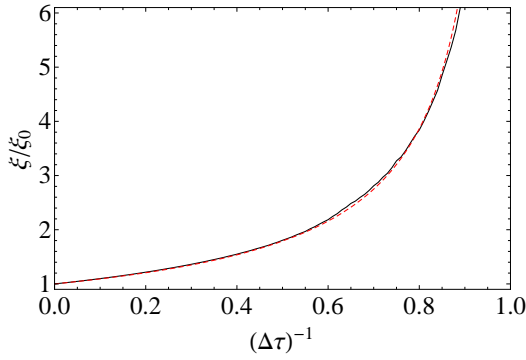


Figure 4. (Color online) Plot of the localization length of the MM as a function of disorder strength. The black solid line shows the numerical values extracted from Fig. 3 by fitting the tails of the curves (at $\log \frac{\nu(x)}{\nu(x=0)} < -6$) with straight lines. Note that the result is meaningful only for weak disorder ($\tau^{-1} \lesssim \Delta$) where the corresponding curve in Fig. 3 is approximately linear. The red dashed line is the best-fit line of a power-law form of $\frac{\xi}{\xi_0} = [1 - (\Delta\tau)^{-1}]^{-0.84}$.

$\sum_n \frac{\psi_n(x)\psi_n^*(x)}{\omega - E_n + i\delta}$ where the summation is over all eigenmodes with energies E_n . Therefore, a localized zero-energy MM with wave function of the form $\sim e^{-x/\xi}$ will result in a decay of the DOS as $\nu(x, \omega = 0) \sim e^{-2x/\xi}$, provided that the bulk gap is finite. Note that it is convenient to ignore the fast-oscillating DOS contributed by g_{23} in Eq. (15).

In Fig. 3 we plot the zero-energy DOS $\nu(x, \omega = 0)$ in log scale, for a range of disorder strength τ^{-1} up to the critical strength where the bulk gap closes. For the clean limit $\tau^{-1} = 0$, the plot is linear with a slope of $\frac{-2}{\xi_0}$, as expected since the MM is localized with a decay length of ξ_0 . When disorder is increased, the slope decreases in magnitude and the curve deviates from a linear behavior. As the strength is increased to the critical gap-closing value ($\tau^{-1} = \Delta$), the decay ceases to be exponential and becomes power-law in nature, as is clear from the linearity of the curve in the log-log plot shown in the inset of Fig. 3. A linear fit through the log-log plot shows that the decay of the ZBP is a power law with a behavior of x^{-1} .

To be more quantitative, the decay length ξ of the Majorana mode could be crudely estimated from Fig. 3, in the weak disorder limit (roughly when $\tau^{-1} \lesssim \Delta$) where the curves are approximately linear, by fitting the curves with straight lines. We compute the slope m of the best-fit line of the tail of each curve in Fig. 3 and extract the estimated decay length ξ of the Majorana mode by $\xi \sim \frac{-2}{m}$, with the results shown in Fig. 4. For the purpose of completeness, Fig. 4 is presented with disorder ranging from zero to the gap-closure limit ($\tau^{-1} = \Delta$), but it should be cautioned that near the gap-closure limit the notion of “decay length” is meaningless as the decay behavior shows a crossover from exponential to power-law. To understand the nature of the divergence at $\tau^{-1} = \Delta$,

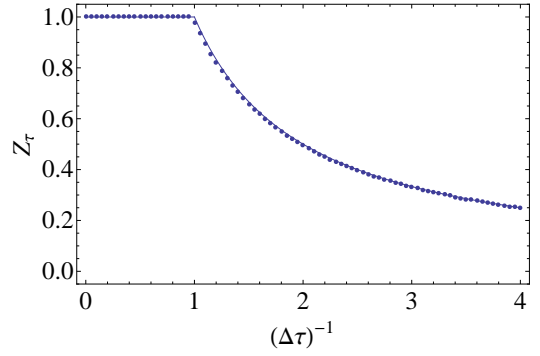


Figure 5. (Color online) Plot of the spectral weight Z_τ (defined in the text) of the zero-energy end mode against disorder strength. The dots show the values obtained from the numerical solution of Eqs. (14) for a range of disorder strengths. The solid line plots the empirical formula Eq. (34).

we fit the curve with a power-law function and obtain $\frac{\xi}{\xi_0} \simeq [1 - (\Delta\tau)^{-1}]^{-0.84}$. Figure 4 shows that this empirical form captures the variations of decay length very well.

VI. LEAKAGE OF THE MAJORANA MODE

The zero-energy MM appears to persist even after the gap closes within our formalism. More precisely, the DOS at the boundary $\nu(x=0, \omega)$ has a pole at $\omega = 0$ for any finite values of Δ and τ . This fact could be derived directly from Eqs. (14a)-(14c) with a perturbative treatment in Δ (see Appendix B). As we know from the case of the clean wire that the divergence at zero-energy comes from a single MM, we fit the DOS near the end of the wire and near zero energy with a Lorentzian form:

$$\nu_\tau(x, \omega, \eta) \sim \frac{1}{2\pi} Z_\tau(x) \frac{\eta}{\omega^2 + \eta^2} + \nu_{\text{reg}}, \quad (32)$$

where $Z_\tau(x)$ is a fitting parameter and the subscript τ indicates the dependence on disorder strength. η is an artificial broadening parameter and ν_{reg} is the part of the DOS that remains non-divergent as $\eta, \omega \rightarrow 0$, contributed from the other delocalized modes in the system.

On the other hand, we know that if the DOS is contributed by a single mode ψ_0 , its exact form is

$$\nu^{(0)}(x, \omega, \eta) = \frac{1}{2\pi} \sum_\lambda |\psi_{0\lambda}(x)|^2 \frac{\eta}{\omega^2 + \eta^2}, \quad (33)$$

where the summation \sum_λ is over the four-component BdG spinor. Comparing Eqs. (32) and (33), it is seen that the spectral weight defined as $Z_\tau = \int_0^\infty Z_\tau(x) dx$ is normalized to unity provided that the MM is not hybridized with other modes.

Figure 5 shows the variations of Z_τ as the strength of disorder is changed. For $\Delta\tau \geq 1$, Z_τ remains around

unity, which is expected as the bulk gap is not closed and the zero-energy MM remains exponentially localized and protected by the spectral gap (and therefore of unit spectral weight). As disorder is increased beyond the strength where the bulk gap closes, Z_τ starts to decrease below unity. This reduction in the spectral weight can be understood as a consequence of the hybridization between the continuum modes in the bulk and the MM. Interestingly, the dependence of Z_τ on disorder can be captured almost perfectly with the empirical formula

$$Z_\tau = \begin{cases} 1, & \Delta\tau \geq 1 \\ \Delta\tau, & \Delta\tau < 1. \end{cases} \quad (34)$$

We note that Eq. (34) indicates a continuous decrease of the MM spectral weight from unity in the topologically gapped situation to a small, but not necessarily vanishingly small, value in the gapless phase. This robustness of the MM spectral weight even in the presence of fairly strong disorder (which completely closes the bulk topological gap) may be the reason for the existence of the ZBP in nanowires which do not necessarily have very high mobilities or obvious superconducting gaps.

VII. CONCLUSION

In this paper we have derived a theory for a disordered p -wave superconductor in 1D, with the effects of disorder incorporated by SCBA. Our theory is thus the p -wave generalization of the Eilenberger theory to 1D systems with the explicit inclusion of disorder. A brief comparison with previous works is in order. Reference 29 applied the Eilenberger equations to a spin-orbit coupled wire with proximity-induced Zeeman term and superconductivity, but the disorder was introduced *after* the Eilenberger equations were obtained and explicit disorder-averaging was performed numerically. Reference 38 adopted the Eilenberger equations to the same system investigated by us, but the emphasis was put on the analysis of proximity effect and *no* disorder was introduced. Moreover, short-length-scale fluctuations in the DOS were explicitly ignored in Reference 38. Our study differs from these works in that disorder is incorporated by SCBA in the Eilenberger equations, and spatial fluctuations of the DOS of the order of Fermi wavelength is retained. In fact, the inclusion of both disorder and spatial fluctuations are the main features of our theory distinguishing it from earlier works in the literature.

We applied our formalism to a semi-infinite p -wave superconducting wire, and found that the gap of the system in the bulk is suppressed by disorder in a way consistent with previous studies. We then focused on the MM located at the end of the wire. We found that with the bulk gap being suppressed, the localization length of the MM increases, and diverges when the gap vanishes. In this process, the localization behavior of the MM changes from exponential to a power-law decay. We

also pointed out an unusual feature of the MM under disorder in this formalism: the DOS shows a divergence at zero-energy at the end of wire even at strong disorder. This is contradictory to the fact that the MM should hybridize with the continuum modes and its spectrum should broaden. However, we can still extract certain manifestations of this hybridization within this formalism—the spectral weight of the MM decreases after the bulk gap is closed, showing a “leakage” of the MM to the continuum. It is interesting that we find that some vestiges (“Majorana ghosts”) of the MMs survive strong disorder and continue showing up in the zero-energy DOS even when the p -wave system has become essentially a gapless system due to disorder.

The results from SCBA appear qualitatively consistent with numerical solutions of the DOS [28] near the end. In these studies the ZBP, which starts as a sharp Majorana peak, decreases in height and broadens out into a peak resulting from Griffiths singularities [18] that is consistent with the class-D symmetry of the system [29]. In contrast to the more exact results where the ZBP is found to broaden into a power-law singularity, we find that the ZBP stays sharp near zero energy while reducing in spectral weight. This discrepancy is not unexpected since the SCBA is a mean-field theory and cannot possibly describe critical fluctuations. Furthermore, we cannot expect to determine a sharp phase transition based on SCBA since SCBA does not describe the localized phase of 1D metals. The disorder-induced topological superconducting phase transition in spinless p -wave superconductors occurs when the superconducting coherence length becomes comparable to the localization length. In summary, SCBA is found to describe qualitatively the suppression of the Majorana ZBP despite the fact that it smears out the phase transition into a crossover from a topological superconducting to a diffusive metallic phase.

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Appendix A: SCBA in a Linearized Model

The self-energy due to ensemble-averaged disorder is

$$\Sigma(x, x') = \delta(x - x') \left\langle V(x) V(x') G^{(0)}(x, x') \right\rangle, \quad (\text{A1})$$

where $G^{(0)}$ is the Green function of the nonlinearized fermion operator. With the linearization $\psi(x) \simeq$

$\psi_R e^{ik_F x} + \psi_L e^{-ik_F x}$, V and $G^{(0)}$ becomes

$$\begin{aligned} G(x, x') &\simeq G_{RR} e^{ik_F(x-x')} + G_{RL} e^{ik_F(x+x')} \\ &\quad + G_{LR} e^{-ik_F(x+x')} + G_{LL} e^{-ik_F(x-x')} \quad (\text{A2}) \\ V(x) &\simeq V_f(x) + V_b(x) e^{2ik_F x} \\ &\quad + V_b^*(x) e^{-2ik_F x}. \quad (\text{A3}) \end{aligned}$$

Using the correlations given in Eqs. (11), the self-energy becomes

$$\begin{aligned} \Sigma(x, x') &= \delta(x - x') \{ D_f G(x, x') \\ &\quad + D_b [G_{RR} e^{ik_F(3x-3x')} + G_{RR} e^{ik_F(-x+x')} + G_{RL} e^{ik_F(3x-x')} G_{RL} e^{ik_F(-x+3x')} \\ &\quad + G_{LR} e^{-ik_F(3x-x')} + G_{LR} e^{-ik_F(-x+3x')} + G_{LL} e^{-ik_F(3x-3x')} + G_{LL} e^{-ik_F(-x+x')}] \}, \quad (\text{A4}) \end{aligned}$$

$$\simeq \delta(x - x') \left\{ D_f G(x, x') + D_b [G_{RR} e^{-ik_F(x-x')} + G_{LL} e^{ik_F(x-x')}] \right\}, \quad (\text{A5})$$

where in the last step only terms proportional to $e^{\pm ik_F x}$ are retained. The linearized same-point self-energy is therefore

$$\Sigma_{RR} = D_f G_{RR} + D_b G_{LL}, \quad (\text{A6a})$$

$$\Sigma_{RL} = D_f G_{RL}, \quad (\text{A6b})$$

$$\Sigma_{LR} = D_f G_{LR}, \quad (\text{A6c})$$

$$\Sigma_{LL} = D_f G_{LL} + D_b G_{RR}. \quad (\text{A6d})$$

When expressed in the chiral Nambu-Gorkov basis $(\psi_R, \psi_L, \psi_L^\dagger, \psi_R^\dagger)$ used in the main text, we have

$$\Sigma = D_f \tau_3 G \tau_3 + \frac{D_b}{2} \tau_3 (\sigma_1 G \sigma_1 + \sigma_2 G \sigma_2) \tau_3. \quad (\text{A7})$$

Appendix B: Singularity of DOS at $(x=0, \omega=0)$ for $\Delta \ll \tau^{-1}$

In the limit $\Delta \ll \tau^{-1}$, we treat Δ as a small parameter and expand the solution to Eq. (14) perturbatively in Δ . For simplicity we shall consider only Eq. (14a-c) supplemented with the boundary conditions Eqs. (16) and Eq. (18a), since the other equations are decoupled and does not affect g_{33} which determines the DOS. At $\Delta = 0$ the problem is trivially solved with

$$g_{33}^{(0)} = 1, \quad (\text{B1a})$$

$$g_{31}^{(0)} = g_{02}^{(0)} = 0. \quad (\text{B1b})$$

With small Δ , we write $g_J = \sum_{n=0}^{\infty} g_J^{(n)} \Delta^n$ (for $J = \{33, 31, 02\}$) and expand Eq. (14) to successive orders in Δ . To the first order in Δ , the system of differential

equations is

$$v_F \partial_x g_{31}^{(1)} = 2\omega g_{02}^{(1)} + 2i + \frac{2i}{\tau} g_{02}^{(1)}, \quad (\text{B2a})$$

$$v_F \partial_x g_{02}^{(1)} = -2\omega g_{31}^{(1)}, \quad (\text{B2b})$$

$$v_F \partial_x g_{33}^{(1)} = 0, \quad (\text{B2c})$$

subjected to the boundary conditions of $g_{02}^{(1)}(0) = 0$ and $\lim_{x \rightarrow \infty} g_{31}^{(1)}(x) = 0$. This is solved with

$$g_{31}^{(1)}(x) = -\frac{e^{-2xi\sqrt{\omega(\omega+i\tau^{-1})}/v_F}}{\sqrt{\omega(\omega+i\tau^{-1})}}, \quad (\text{B3a})$$

$$g_{02}^{(1)}(x) = \frac{ie^{-2xi\sqrt{\omega(\omega+i\tau^{-1})}/v_F}}{\omega+i\tau^{-1}} - \frac{i}{\omega+i\tau^{-1}}, \quad (\text{B3b})$$

$$g_{33}^{(1)}(x) = 0, \quad (\text{B3c})$$

which has no effect on the DOS. We must therefore go to the second order which gives

$$v_F \partial_x g_{33}^{(2)} = -2ig_{31}^{(1)} - \frac{2i}{\tau} g_{31}^{(1)} g_{02}^{(1)}, \quad (\text{B4a})$$

where only the equation for $g_{33}^{(2)}$ is given as it is relevant to the evaluation to DOS. Requiring $\lim_{x \rightarrow \infty} g_{33}^{(2)}(x) = \frac{1}{2(\omega+i\tau^{-1})^2}$ which follows from the expansion of Eq. (28c), we have

$$\begin{aligned} g_{33}^{(2)}(x) &= -\frac{ie^{-4ix\sqrt{\omega(\omega+i\tau^{-1})}/v_F}}{2\omega\tau(\omega+i\tau^{-1})^2} - \frac{e^{-2ix\sqrt{\omega(\omega+i\tau^{-1})}/v_F}}{(\omega+i\tau^{-1})} \\ &\quad + \frac{1}{2(\omega+i\tau^{-1})^2}, \quad (\text{B5}) \end{aligned}$$

$$g_{33}^{(2)}(0) \approx \frac{i\tau}{2\omega} - \frac{\tau^2}{2} - \frac{i\omega\tau^3}{2}, \quad (\text{B6})$$

in which an expansion in ω is performed. We therefore see that the pole at zero energy is present even for $\Delta\tau \ll 1$.

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